Math 347 Worksheet

Lecture 12: Operations on functions

September 26, 2018

For any function $f : \mathbb{R} \to \mathbb{R}$, given any real number $a \in \mathbb{R}$ we can define the *translation of f by a* as the following function

$$\begin{aligned} T_a f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x+a). \end{aligned}$$

Similarly, for any number $b \in \mathbb{R}$ one defines *the scaling of f by b* to be the function

$$\begin{aligned} S_b f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(b\,x). \end{aligned}$$

1) For given $a, b \in \mathbb{R}$, can you describe the operations T_a and S_b as functions? What is the domain and codomain of those functions?

2) For the sets that you considered in 1) can you think of another function defined on those sets?

3) Is the operation T_a seen as a function injective? Is it surjective? What about S_b?

4) Suppose $f(x)$ is a polynomial function. Prove that $T_a f(x) = f(x)$ for all $x, a \in \mathbb{R}$ if and only if $f(x)$ is constant.

5) Suppose $f(x)$ is a polynomial function. Prove that $S_b f(x) = b f(x)$ for all $x, a \in \mathbb{R}$ if and only if $f(x) = cx$ for some $c \in \mathbb{R}$. Notice that the same holds if one requires that $T_a f(x) = f(x) + f(a)$ for all $a, x \in \mathbb{R}$ instead.

6) For which functions f is $T_a S_b f(x) = S_b T_a f(x)$ for all $x, a, b \in \mathbb{R}$?

7) Let A and B be finite and disjoint sets, prove that

$$|A| + |B| = |A \cup B|.$$

Recall that $|A|$ is defined as the unique $n \geq 0$ such that there exists a bijection between A and $\{1, \ldots, n\}$.

8) Two sets A and B are said to have the same *cardinality* if there exists a bijection between them. Prove that $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ have the same cardinality.

9) Consider the function $f : (0,1) \to \mathbb{R}$ given by $f(x) = \tan(\pi(x - 1/2))$. What can you conclude about the cardinality of $(0,1)$ and $\mathbb{R}$?

10) Do $\mathbb{N}$ and $P(\mathbb{N})$ have the same cardinality?